I noticed in my mathematical meanderings involving Phi I often had to reduce algebraic formulae involving powers of Phi. I began to wonder if there was an elegant way to represent those powers that is both useful and simple. I first began with the positive integer powers, 1, 2, 3, ….

$$\phi^0 = 1$$
$$\phi^1 = \tfrac{1}{2}[\sqrt{5} + 1]$$
$$\phi^2 = \tfrac{1}{4}[2\sqrt{5} + 6] = \tfrac{1}{2}[\sqrt{5} + 3]$$
$$\phi^3 = \tfrac{1}{4}[4\sqrt{5} + 8] = \tfrac{1}{2}[2\sqrt{5} + 4]$$
$$\phi^4 = \tfrac{1}{2}[3\sqrt{5} + 7]$$
$$\phi^5 = \tfrac{1}{4}[10\sqrt{5} + 22] = \tfrac{1}{2}[5\sqrt{5} + 11]$$
$$\phi^6 = \tfrac{1}{4}[16\sqrt{5} + 36] = \tfrac{1}{2}[8\sqrt{5} + 18]$$
$$\phi^7 = \tfrac{1}{2}[13\sqrt{5} + 29]$$
$$\phi^8 = \tfrac{1}{4}[42\sqrt{5} + 94] = \tfrac{1}{2}[21\sqrt{5} + 47]$$
$$\phi^9 = \tfrac{1}{4}[68\sqrt{5} + 152] = \tfrac{1}{2}[34\sqrt{5} + 76]$$

And so I noticed the Fibonacci series as the first coefficient of the radical easily enough, and I speculated about the second term for a short bit, quickly verifying that it is indeed consecutive elements in the Lucas series. Thus, a nice way to represent the general form for all non-negative integer powers of Phi yields:

$$\phi^n = \tfrac{1}{2}[F_{(n)}\sqrt{5} + L_{(n)}],\ n \geq 0 \qquad (1.1)$$

Nice! Further analysis of the negative powers followed thus:

$$\phi^{-1} = \phi - 1 = \tfrac{1}{2}[\sqrt{5} - 1]$$
$$\phi^{-2} = \tfrac{1}{2}[3 - \sqrt{5}]$$
$$\phi^{-3} = \tfrac{1}{4}[4\sqrt{5} - 8] = \tfrac{1}{2}[2\sqrt{5} - 4]$$
$$\phi^{-4} = \tfrac{1}{2}[7 - 3\sqrt{5}]$$
$$\phi^{-5} = \tfrac{1}{2}[5\sqrt{5} - 11]$$

The pattern here was quickly becoming apparent, resulting in the following for all non-positive integers n:

$$\phi^n = \tfrac{1}{2}[(-1)^{n-1}F_{(|n|)}\sqrt{5} + (-1)^n L_{(|n|)}],\ n \leq 0 \qquad (1.2)$$

Combining the two resulted in the following relation over all integers n:

$$\phi^n = \tfrac{1}{2}[(-1)^{(a)}F_{(|n|)}\sqrt{5} + (-1)^{(b)} L_{(|n|)}] \qquad (1.3)$$

where:

n	even and negative	odd and negative	0	even and positive	odd and positive
a	1	0	0	0	0
b	0	1	0	0	0

Or, $a = n - 1$, $b = n$ when n is negative, $a = b = 0$ all other n.

In all of the above formulae $F_{(n)}$ represents the n^{th} member of the Fibonacci series, where $F_{(0)} = 0$, $F_{(1)} = 1$, and $F_{(n+2)} = F_{(n)} + F_{(n+1)}$. Likewise $L_{(n)}$ denotes the n^{th} member of the Lucas series, where $L_{(0)} = 2$, $L_{(1)} = 1$, and $L_{(n+2)} = L_{(n)} + L_{(n+1)}$.

Some interesting twists on the above give the following Fibonacci and Lucas relationships:

$$F_{(n)} = (2\phi^n - L_{(n)})/\sqrt{5} \qquad (1.4)$$

and

$$L_{(n)} = 2\phi^n - F_{(n)}\sqrt{5} \qquad (1.5)$$

I have not seen these five relations published, either on the Internet or on paper, having read a dozen or so books on the subject. I have not gotten hold of S. Vajda's "*Fibonacci and Lucas Numbers, and the Golden Section*", Ellis Horwood Limited, Chichester, 1989. I have a feeling they may be printed there, but until such time as I can verify, as the publication is currently out of print, I hope these will stand as my contributions.